Lightning talk YGGT

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Ultrametric analogues of Ulam stability

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General framework

$d_G(kg, kh) = d_G(g,h) = d_G(gk, hk)$

- $\mathcal{G}$ a family of groups G equipped with bi-invariant metrics d_G,
  Γ a countable group.

Q (Stability): of Γ wrt $\mathcal{G}$

$$\boxed{\begin{array}{l}\text{Is every almost-homomorphism } \underline{\varphi : \Gamma \to G \in \mathcal{G}} \\ \text{close to a homomorphism } \underline{\psi : \Gamma \to G} \text{ ?}\end{array}}$$

- Formalized using:

$$\operatorname{def}_{g,h}(\varphi) := d_G(\varphi(gh), \varphi(g)\varphi(h))$$

$$\operatorname{def}(\varphi) := \sup_{g,h} \operatorname{def}_{g,h}(\varphi)$$

$$\operatorname{dist}_g(\varphi, \psi) := d_G(\varphi(g), \psi(g))$$

$$\operatorname{dist}(\varphi, \psi) := \sup_g \operatorname{dist}_g(\varphi, \psi)$$

$\rightsquigarrow$ pointwise stability / uniform stability

- Most commonly studied:

Hamming distance

- $\mathcal{G} = (S_n, d_{Hamm})$, mostly pointwise

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some bi-invariant norm

- $\mathcal{G} = (U(n), \|\cdot\|)$

Quite different behaviours and techniques!

IRS's

Theoretical CS

(Bounded) cohomology

C^*-algebras

K-theory

! Pointwise stability $\underset{\not\Leftarrow}{\not\Rightarrow}$ uniform stability

EX: $\mathcal{G} = (U(n), \|\cdot\|_{op})$. Then F_2 is pointwise, not uniformly stable
$\mathbb{Z}^2$ is uniformly, not pointwise stable

Ultrametric families

Q: What happens if every $G \in \mathcal{G}$ is ultrametric? That is:

$$\boxed{d_G(g,h) \leq \max\{d_G(g,k),\ d_G(k,h)\}} \quad \forall g,h,k \in G$$

Typical example: 1^{st}-countable profinite group with metric induced by a filtration

EX: - Families of Galois groups
- Families of groups of rooted tree automorphism
- (Main) $GL_n(\mathbb{Z}_p)$, $GL_n(\mathbb{F}_q[[X]])$
= non-Archimedian analogues of $U(n)$

Pointwise vs uniform stability

Pointwise vs uniform stability

THM: Let Γ be finitely generated. Then
1) Γ is pointwise stable $\Rightarrow$ Γ is uniformly stable.
2) If Γ is finitely presented, then $\Leftarrow$.

EX: Free groups are uniformly stable wrt ultrametric families.
They never are in the other examples!

Stability via finite quotients

- Suppose now the groups in $\mathcal{G}$ are moreover compact. Then:

THM: Γ is uniformly stable $\Leftrightarrow$ its largest residually finite

THM: Γ is uniformly stable $\iff$ its largest residually finite quotient is

Major open questions in some other settings!

EX: 1) Groups without finite quotients are uniformly stable.

2) H a perfect group $\rightsquigarrow$ $H \wr \mathbb{Z}$ is uniformly stable.

- For pointwise stability, residual properties are replaced by local embeddings

Concrete examples

- $\mathcal{G} = GL_n(\mathbb{Z}_p)$ or $GL_n(\mathbb{F}_q[[X]])$ $q = p^k$

THM: The following are uniformly stable

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1) Virtually free groups w/o p-torsion
2) $BS(m,n)$ where $p \mid m$, $p \nmid n$ ← generalizes to GBS groups
3) $\mathbb{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbb{Z}$ if moreover $1 \neq |m| \neq |n| \neq 1$, $(m,n)=1$
4) $H \wr \mathbb{Z}$ if $H \not\twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$

- In characteristic 0, more tools are available.
  $\mathcal{G} = GL_n(\mathbb{Z}_p)$.

THM: The following are uniformly stable:

1) Finite groups
2) Virtually free groups
3) Finitely generated virtually free groups

generalizes

3) Finitely generated virtually free groups

4) $BS(m,n)$ where $\nu_p(m) \neq \nu_p(n)$. generalizes to GBS groups

Open questions

1) Is $\mathbb{Z}^2$ $\mathcal{G}$-stable, for $\mathcal{G} = GL_n(\mathbb{Z}_p)$ or $GL_n(\mathbb{F}_q[[X]])$

2) Are all finite groups $\mathcal{G}$-stable, for $\mathcal{G} = GL_n(\mathbb{F}_q[[X]])$?